On generalization of dual Fibonacci octonions

Serpil Halici and Adnan Karataş*

Pamukkale University, Faculty of Arts and Sciences, Department of Mathematics, Denizli/TURKEY
*Corresponding Author
E-mail: shalici@pau.edu.tr, adnank@pau.edu.tr

Abstract

In this study, we examine all the second order linear recurrence relations over dual octonions. Hence, we generalize Fibonacci-like relations over dual octonions. For this purpose we use the well-known Horadam sequence and obtain some fundamental and new identities involving elements of this generalized sequence.

2010 Mathematics Subject Classification. **11B39** 17A20
Keywords. Fibonacci numbers and generalization, Horadam sequence, octonions.

1 Introduction

A dual octonion can be defined as $\hat{K} = (k, l) = k + l\varepsilon$ where $k, l \in \mathbb{O}$. Let $\mathbb{D}$ be the set of dual octonions;

$$\mathbb{D} = \{\hat{K} | \hat{K} = (k, l) = k + l\varepsilon; \quad k, l \in \mathbb{O} \text{ with } \varepsilon^2 = 0\}. \tag{1.1}$$

The addition on the set $\mathbb{D}$ is given by

$$\hat{K} + \hat{L} = (k, l) + (m, n) = (k + m, l + n) = (k + m) + (l + n)\varepsilon. \tag{1.2}$$

It should be noted that there are different multiplication tables for octonions which can be obtained by Cayley-Dickson process [4, 5]. We define the multiplication of the bases elements as $e_i e_j = e_k$. The set of indices $\{i, j, k\}$ being one of $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{1, 7, 6\}$, $\{2, 4, 6\}$, $\{2, 5, 7\}$, $\{3, 4, 7\}$ and $\{3, 6, 5\}$. Note that if a different multiplication is chosen, then one should make all calculations accordingly (see, [1, 4]). The multiplication of two dual octonions can be defined as

$$\widehat{K}\widehat{L} = (km, kn + lm). \tag{1.3}$$

The conjugate of a dual octonion $\widehat{K}$ is going to be denoted by $\overline{\widehat{K}}$ and this will be written as follows.

$$\overline{\widehat{K}} = (\overline{k}, \overline{l}) = \overline{k} + \overline{l}\varepsilon. \tag{1.4}$$

Norm is defined by the help of conjugate as follows;

$$N(\widehat{K}) = (k, l)(\overline{k}, \overline{l}) = (\overline{k}, \overline{l})(k, l) = (k\overline{k}, k\overline{l} + l\overline{k}). \tag{1.5}$$

We also note that the definition of a dual unit ε causes the norm to be degenerate which means that the norm can be equal to zero for some non zero dual octonions. To explain this situation let us calculate the norm for $\widehat{K} = 1\varepsilon$;

$$N(\widehat{K}) = 1\varepsilon\overline{1}\varepsilon = 1\varepsilon^2 = 0. \tag{1.6}$$

Tbilisi Mathematical Journal Special Issue (IECMSA-2019), pp. 59–67.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 08 October 2019.
Accepted for publication: 02 January 2020.

For detailed knowledge, we refer the interested readers to [1] and [3].

The rest of this paper is organized as follows. Section 2 reviews basic properties of Horadam sequence which will be used in the following sections. In Section 3, the dual Horadam octonions are introduced and investigated. Moreover, some important identities and equalities involving these octonions are obtained. And finally, in Section 4, the generalization is made in this study is summarized.

2 The Horadam sequence

In this section, we focus on the second order recursive relations over the dual octonions. There are some papers about this subject in literature. For example, in [6], the author studied dual Fibonacci octonions. In [2], the authors investigated the dual k-Pell octonions. Our aim is to generalize all second order recursive relations over the dual octonions. To achieve this goal, we shall first give some fundamental properties of the Horadam sequence. The Horadam sequence is a particular type of linear recurrence sequences. This sequence generalizes all of the well-known sequences such as Fibonacci, Lucas, Pell, Pell-Lucas sequences, etc. Some studies containing the Horadam sequence can be seen in the references [8–10].

Let us start with the definition of Horadam sequence. In [9], Horadam gave the following sequence definition.

$$\{w_n(a,b;p,q)\}; \quad n \geq 2 \tag{2.1}$$

where

$$w_n = pw_{n-1} + qw_{n-2}; \quad w_0 = a, w_1 = b \tag{2.2}$$

If one writes $a = 0, b = p = q = 1$ in the equation (2.2), then the classical Fibonacci sequence is obtained. Also, the other types of sequences can be obtained such as Lucas, Pell, Pell-Lucas. Now, we present some formulas which will be used very frequently. It is known one of the fundamental identity related to the recurrence relations is the Binet formula, and it can be calculated by using the characteristic equation. For the Horadam sequence, the roots of the characteristic equation are

$$\alpha = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad \beta = \frac{p - \sqrt{p^2 + 4q}}{2}. \tag{2.3}$$

Using these roots and the recurrence relation the Binet formula can be given as follows;

$$w_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \qquad A = b - a\beta \text{ and } B = b - a\alpha. \tag{2.4}$$

We will examine Binet formula, Cassini identity, d'Ocagne's identity and other equalities for the dual Horadam octonions. It is worth noting that there are some studies related to the second order recurrence relations over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ in literature. One of the earliest studies on this topic belongs to Horadam where it was given a generalization of the second order recurrence relations on $\mathbb{R}$ and $\mathbb{C}$. In [12], Swamy defined the generalized Fibonacci quaternions and then, in [7], we defined quaternions with coefficients from Horadam sequence. Furthermore, in [11], we described and studied the Horadam octonions. And for these octonions, we obtained the Binet formula and some important identities.

3 Dual Horadam octonions

In this section, we define the dual Horadam octonions and give the identities and equalities mentioned in Section 2. Furthermore, we show that the identities involving dual Horadam octonions can be reduced to the identities which are obtained by using different initial values in [6, 13].

Now, let us define a new type of octonions which we call as dual Horadam octonions. Note that these octonions are dual octonions which have coefficients from Horadam sequence. In the rest of the study we will use the notation $\widehat{\mathbb{O}G_n}$ to demonstrate the *n-th* dual Horadam octonion. Let us define the *n-th* dual Horadam octonion as follows.

$$\widehat{\mathbb{O}G_n} = (\mathbb{O}G_n, \mathbb{O}G_{n+1}) = \mathbb{O}G_n + \mathbb{O}G_{n+1}\varepsilon \tag{3.1}$$

where

$$\mathbb{O}G_n = w_n e_0 + w_{n+1} e_1 + \cdots + w_{n+7} e_7. \tag{3.2}$$

Firstly, we give some fundamental properties of these dual octonions. For one of the main features of the dual Horadam octonions we give Binet formula in the following theorem.

Theorem 3.1. For $n \geq 1$, we get

$$\widehat{\mathbb{O}G_n} = \frac{A\underline{\alpha}\alpha^n(1+\alpha\varepsilon) - B\underline{\beta}\beta^n(1+\beta\varepsilon)}{\alpha-\beta}, \tag{3.3}$$

where $\underline{\alpha} = 1e_0 + \alpha e_1 + \alpha^2 e_2 + \cdots + \alpha^7 e_7$ and $\underline{\beta} = 1e_0 + \beta e_1 + \beta^2 e_2 + \cdots + \beta^7 e_7$.

Proof. In [11], the Binet formula for Horadam octonions is given as follows.

$$\mathbb{O}G_n = \frac{A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n}{\alpha-\beta} \tag{3.4}$$

where

$$\underline{\alpha} = 1e_0 + \alpha e_1 + \alpha^2 e_2 + \cdots + \alpha^7 e_7 \text{ and } \underline{\beta} = 1e_0 + \beta e_1 + \beta^2 e_2 + \cdots + \beta^7 e_7. \tag{3.5}$$

For the dual Horadam octonions Binet formula can be obtained in a similar way. Using the definition of dual Horadam octonions and its characteristic equation

$$\widehat{\mathbb{O}G_n} = \frac{A\underline{\alpha}\alpha^n(1+\alpha\varepsilon) - B\underline{\beta}\beta^n(1+\beta\varepsilon)}{\alpha-\beta}$$

is obtained which is desired. Where

$$A = b - a\beta, \quad B = b - a\alpha$$

and

$$\alpha = \frac{p+\sqrt{p^2+4q}}{2}, \quad \beta = \frac{p-\sqrt{p^2+4q}}{2}.$$

Q.E.D.

One can realize that the equation (3.3) generalizes the Binet formulas which are obtained in [6,13]. That is, for α, β and $A = B = 1$, the formula (3.3) is reduced to following equation.

$$\frac{\underline{\alpha}\alpha^n(1+\alpha\varepsilon) - \underline{\beta}\beta^n(1+\beta\varepsilon)}{\alpha-\beta}.$$

In the next theorem, we give the generating function for dual Horadam octonions.

Theorem 3.2. The generating function for the dual Horadam octonions is

$$\frac{\widehat{\mathbb{O}G_0} + (\widehat{\mathbb{O}G_1} - p\widehat{\mathbb{O}G_0})t}{1-pt-qt^2}. \tag{3.6}$$

Proof. To prove this claim, firstly we write $g(t)$ as

$$g(t) = \widehat{\mathbb{O}G_0}t^0 + \widehat{\mathbb{O}G_1}t + \cdots + \widehat{\mathbb{O}G_n}t^n + \dots. \tag{3.7}$$

Secondly, we need to calculate $ptg(t)$ and $qt^2g(t)$ in the following equations;

$$ptg(t) = \sum_{n=0}^{\infty} p\widehat{\mathbb{O}G_n}t^{n+1} \text{ and } qt^2g(t) = \sum_{n=0}^{\infty} q\widehat{\mathbb{O}G_n}t^{n+2}. \tag{3.8}$$

Hence, after straightforward calculations, we obtain

$$g(t) = \frac{\mathbb{O}G_0 + \mathbb{O}G_1\varepsilon + (\mathbb{O}G_1 + \mathbb{O}G_2\varepsilon - p(\mathbb{O}G_0 + \mathbb{O}G_1\varepsilon))t}{1-pt-qt^2}, \tag{3.9}$$

which is the generating function for the dual Horadam octonions. Q.E.D.

We want to note that the formula (3.6) generalizes the next formula which is given in [6].

$$\frac{\widehat{\mathbb{O}_0} + (\widehat{\mathbb{O}_1} - \widehat{\mathbb{O}_0})t}{1-t-t^2}. \tag{3.10}$$

For the dual Fibonacci and dual Lucas octonions, Cassini identities are studied in [6,13]. In the following theorem, we give the Cassini formula for dual Horadam octonions.

Theorem 3.3. For dual Horadam octonions, we have

$$\widehat{\mathbb{O}G_{n-1}}\widehat{\mathbb{O}G_{n+1}} - \widehat{\mathbb{O}G_n^2} = \frac{AB(-q)^{n-1}\underline{\alpha}\underline{\beta}[p+(p^2+q)\varepsilon]}{\sqrt{p^2+4q}}. \tag{3.11}$$

Proof. For the proof equation (3.11) if we use the Binet formula, then we have

$$\Big(\frac{A\underline{\alpha}\alpha^{n-1}(1+\alpha\varepsilon) - B\underline{\beta}\beta^{n-1}(1+\beta\varepsilon)}{\alpha-\beta}\Big)\Big(\frac{A\underline{\alpha}\alpha^{n+1}(1+\alpha\varepsilon) - B\underline{\beta}\beta^{n+1}(1+\beta\varepsilon)}{\alpha-\beta}\Big)$$

$$-\Big(\frac{A\underline{\alpha}\alpha^{n}(1+\alpha\varepsilon) - B\underline{\beta}\beta^{n}(1+\beta\varepsilon)}{\alpha-\beta}\Big)^2.$$

After making necessary calculations

$$\frac{AB(-q)^{n-1}\underline{\alpha}\underline{\beta}[(\alpha^2-\beta^2)+(\alpha^3-\beta^3)\varepsilon]}{(\alpha-\beta)^2}$$

is obtained which can be rearranged as,

$$\frac{(b^2-abp-qa^2)(-q)^{n-1}\underline{\alpha}\underline{\beta}[p+(p^2+q)\varepsilon]}{\sqrt{p^2+4q}}.$$

Q.E.D.

It should be noted that, in equation (3.11), if we take the necessary initial values then we have a formula which is equivalent to the Cassini formula is given in the reference [6]. Also, according to $\underline{\alpha}$ and $\underline{\beta}$ values for real quaternions, we have Cassini identity stated in [7].

Now, we give d'Ocagne's identity for dual Horadam octonions.

Theorem 3.4. For the dual Horadam octonions, we have

$$\widehat{\mathbb{O}G_{m+1}}\widehat{\mathbb{O}G_n}-\widehat{\mathbb{O}G_m}\widehat{\mathbb{O}G_{n+1}}=\frac{AB\underline{\alpha}\underline{\beta}(\alpha^n\beta^m-\alpha^m\beta^n)[1+p\varepsilon]}{\sqrt{p^2-4q}}. \tag{3.12}$$

Proof. We use the Binet formula in (3.12). Then the right hand side is follows;

$$=\Big(\frac{A\underline{\alpha}\alpha^{m+1}(1+\alpha\varepsilon)-B\underline{\beta}\beta^{m+1}(1+\beta\varepsilon)}{\alpha-\beta}\Big)\Big(\frac{A\underline{\alpha}\alpha^{n}(1+\alpha\varepsilon)-B\underline{\beta}\beta^{n}(1+\beta\varepsilon)}{\alpha-\beta}\Big)$$

$$-\Big(\frac{A\underline{\alpha}\alpha^{m}(1+\alpha\varepsilon)-B\underline{\beta}\beta^{m}(1+\beta\varepsilon)}{\alpha-\beta}\Big)\Big(\frac{A\underline{\alpha}\alpha^{n+1}(1+\alpha\varepsilon)-B\underline{\beta}\beta^{n+1}(1+\beta\varepsilon)}{\alpha-\beta}\Big).$$

After needed calculations,

$$\widehat{\mathbb{O}G_{m+1}}\widehat{\mathbb{O}G_n}-\widehat{\mathbb{O}G_m}\widehat{\mathbb{O}G_{n+1}}=\frac{AB\underline{\alpha}\underline{\beta}(\alpha^n\beta^m-\alpha^m\beta^n)[1+p\varepsilon]}{\sqrt{p^2-4q}}$$

is obtained where $AB=b^2-abp-qa^2$. Q.E.D.

Note that if we focus on non-dual part of the equation (3.12) we get d'Ocagne's identity for real Horadam octonions.

Remark 3.5. For real Horadam octonions, d'Ocagne's identity is as follows;

$$\mathbb{O}G_{m+1}\mathbb{O}G_n-\mathbb{O}G_m\mathbb{O}G_{n+1}=\frac{AB\underline{\alpha}\underline{\beta}(\alpha^n\beta^m-\alpha^m\beta^n)}{\alpha-\beta}.$$

In the following theorem, we give Vajda identity for dual Horadam octonions.

Theorem 3.6. For the dual Horadam octonions, we have

$$\widehat{\mathbb{O}G_{n+m}}\widehat{\mathbb{O}G_{n+k}} - \widehat{\mathbb{O}G_{n}}\widehat{\mathbb{O}G_{n+m+k}} =$$
$$\frac{AB\underline{\alpha\beta}(-q)^n\beta^k(\beta^k-\alpha^m)+(\beta^{m+1}+\alpha\beta^m-\alpha^{m+1}-\alpha^m\beta)\varepsilon}{(\alpha-\beta)^2}+$$
$$\frac{AB\underline{\beta\alpha}(-q)^n\alpha^k(\alpha^k-\beta^m)+(\alpha^{m+1}+\alpha^m\beta-\beta^{m+1}-\beta^m\alpha)\varepsilon}{(\alpha-\beta)^2}. \tag{3.13}$$

Proof. We use the equation (3.3) and rewrite the equation (3.13) explicitly as follows

$$= \Big(\frac{A\underline{\alpha}\alpha^{n+m}(1+\alpha\varepsilon)-B\underline{\beta}\beta^{n+m}(1+\beta\varepsilon)}{\alpha-\beta}\Big)\Big(\frac{A\underline{\alpha}\alpha^{n+k}(1+\alpha\varepsilon)-B\underline{\beta}\beta^{n+k}(1+\beta\varepsilon)}{\alpha-\beta}\Big)$$
$$-\Big(\frac{A\underline{\alpha}\alpha^{n}(1+\alpha\varepsilon)-B\underline{\beta}\beta^{n}(1+\beta\varepsilon)}{\alpha-\beta}\Big)\Big(\frac{A\underline{\alpha}\alpha^{n+m+k}(1+\alpha\varepsilon)-B\underline{\beta}\beta^{n+m+k}(1+\beta\varepsilon)}{\alpha-\beta}\Big).$$

After needed calculations Vajda identity is

$$\widehat{\mathbb{O}G_{n+m}}\widehat{\mathbb{O}G_{n+k}} - \widehat{\mathbb{O}G_{n}}\widehat{\mathbb{O}G_{n+m+k}} =$$
$$\frac{AB\underline{\alpha\beta}\alpha^n\beta^{n+k}(\beta^k-\alpha^m)+(\beta^{m+1}+\alpha\beta^m-\alpha^{m+1}-\alpha^m\beta)\varepsilon}{(\alpha-\beta)^2}+$$
$$\frac{AB\underline{\beta\alpha}\alpha^{n+k}\beta^n(\alpha^k-\beta^m)+(\alpha^{m+1}+\alpha^m\beta-\beta^{m+1}-\beta^m\alpha)\varepsilon}{(\alpha-\beta)^2}$$

where $AB = b^2 - abp - qa^2$. Q.E.D.

For the real Horadam octonions we give following remark.

Remark 3.7. For the real Horadam octonions the Vajda identity is

$$\mathbb{O}G_{n+m}\mathbb{O}G_{n+k} - \mathbb{O}G_n\mathbb{O}G_{n+m+k} = \frac{AB\underline{\alpha\beta}(-q)^n(\beta^{2k}+\beta^{n+m}-\alpha^{2k}-\alpha^{m+k})}{(\alpha-\beta)^2}. \tag{3.14}$$

Theorem 3.8. For dual Horadam octonions, we have the following relations:

$$i)\ \widehat{\mathbb{O}G_{n}}\widehat{\mathbb{O}G_{n+3}} - \widehat{\mathbb{O}G_{n+1}}\widehat{\mathbb{O}G_{n+2}} = -AB\underline{\alpha\beta}(-q)^n[p+(p^2+2q)\varepsilon]. \tag{3.15}$$

$ii)$ $\widehat{\mathbb{O}G_{n}^2} \pm \widehat{\mathbb{O}G_{n+1}^2} =$

$$A^2\underline{\alpha}^2\alpha^{2n}[(1\pm\alpha^2)+2(\alpha\pm\alpha^3)\varepsilon]+B^2\underline{\beta}^2\beta^{2n}[(1\pm\beta^2)+2(\beta\pm\beta^3)\varepsilon]. \tag{3.16}$$

Proof. Let us prove the equation (3.16) with plus sign which is

$$\widehat{\mathbb{O}G_{n}^2} + \widehat{\mathbb{O}G_{n+1}^2} = A^2\underline{\alpha}^2\alpha^{2n}[(1+\alpha^2)+2(\alpha+\alpha^3)\varepsilon]+B^2\underline{\beta}^2\beta^{2n}[(1+\beta^2)+2(\beta+\beta^3)\varepsilon].$$

In order to prove we use the equation (3.3) in the above equation as

$$\Big(\frac{A\underline{\alpha}\alpha^{n}(1+\alpha\varepsilon)-B\underline{\beta}\beta^{n}(1+\beta\varepsilon)}{\alpha-\beta}\Big)^2+\Big(\frac{A\underline{\alpha}\alpha^{n+1}(1+\alpha\varepsilon)-B\underline{\beta}\beta^{n+1}(1+\beta\varepsilon)}{\alpha-\beta}\Big)^2.$$

After using properties of α and β we get

$$A^2\underline{\alpha}^2\alpha^{2n}[(1+\alpha^2)+2(\alpha+\alpha^3)\varepsilon]+B^2\underline{\beta}^2\beta^{2n}[(1+\beta^2)+2(\beta+\beta^3)\varepsilon]$$

which is desired. Q.E.D.

In analogy with the Theorem 3 we state following remark.

Remark 3.9. For real Horadam octonions we have the following identities

$$i)\ \mathbb{O}G_n\mathbb{O}G_{n+3}-\mathbb{O}G_{n+1}\mathbb{O}G_{n+2}=-AB\underline{\alpha}\underline{\beta}(-q)^np. \tag{3.17}$$

$$ii)\ \mathbb{O}G_n^2\pm\mathbb{O}G_{n+1}^2=A^2\underline{\alpha}^2\alpha^{2n}(1\pm\alpha^2)+B^2\underline{\beta}^2\beta^{2n}(1\pm\beta^2). \tag{3.18}$$

In the following theorem, we give a formula which gives the sum of dual Horadam octonions

Theorem 3.10. The summation formula for the dual Horadam octonions is

$$\sum_{i=0}^{n}\widehat{\mathbb{O}G_i}=\sum_{i=0}^{n}\mathbb{O}G_i+\sum_{i=1}^{n+1}\mathbb{O}G_i\varepsilon=(d_1,d_2), \tag{3.19}$$

where the real and dual parts of equation (3.19) are as follows:

$$d_1=\sum_{i=0}^{n}\mathbb{O}G_i=\frac{1}{\alpha-\beta}\Big(\frac{B\underline{\beta}\beta^{n+1}}{1-\beta}-\frac{A\underline{\alpha}\alpha^{n+1}}{1-\alpha}\Big)+K \tag{3.20}$$

and

$$d_2=\sum_{i=1}^{n+1}\mathbb{O}G_i=\frac{1}{\alpha-\beta}\Big(\frac{B\underline{\beta}\beta^{n+2}}{1-\beta}-\frac{A\underline{\alpha}\alpha^{n+2}}{1-\alpha}\Big)-\mathbb{O}G_0+K \tag{3.21}$$

respectively. The value K is

$$K=\frac{A\underline{\alpha}(1-\beta)-B\underline{\beta}(1-\alpha)}{(\alpha-\beta)(1-\alpha)(1-\beta)}. \tag{3.22}$$

Proof. For the summation formula of $\widehat{\mathbb{O}G_n}$ we calculate the values d_1 and d_2 separately. To find d_1 we can use Binet formula and definition of geometric series as follows:

$$d_1=\sum_{i=0}^{n}\mathbb{O}G_i=\frac{A\underline{\alpha}\alpha^n-B\underline{\beta}\beta^n}{\alpha-\beta}=\frac{A\underline{\alpha}}{\alpha-\beta}\sum_{i=0}^{n}\alpha^n-\frac{B\underline{\beta}}{\alpha-\beta}\sum_{i=0}^{n}\beta^n.$$

$$=\frac{A\underline{\alpha}}{\alpha-\beta}\sum_{i=0}^{n}\frac{1-\alpha^{n+1}}{1-\alpha}-\frac{B\underline{\beta}}{\alpha-\beta}\sum_{i=0}^{n}\frac{1-\beta^{n+1}}{1-\beta}.$$

Upon making the necessary calculations, we get the explicit form of d_1 as

$$\sum_{i=0}^{n}\mathbb{O}G_i=\frac{1}{\alpha-\beta}\Big(\frac{B\underline{\beta}\beta^{n+1}}{1-\beta}-\frac{A\underline{\alpha}\alpha^{n+1}}{1-\alpha}\Big)+\frac{A\underline{\alpha}(1-\beta)-B\underline{\beta}(1-\alpha)}{(\alpha-\beta)(1-\alpha)(1-\beta)}.$$

On the other hand, the value of d_2 can easily be calculated similar to d_1. Q.E.D.

We note that the formula in the equation (3.19) generalizes the formula is given by Halici in [6], which is for dual Fibonacci octonions;

$$\sum_{i=1}^{n} \widehat{\mathbb{O}_i} = \widehat{\mathbb{O}_2} F_{n+1} + \widehat{\mathbb{O}_1} F_{n-1} - \widehat{\mathbb{O}_2}. \tag{3.23}$$

Theorem 3.11. The norm of *n-th* dual Horadam octonion is

$$Nr(\widehat{\mathbb{O}G_n}) = (\mathbb{O}G_n \overline{\mathbb{O}G_n},\ \ \mathbb{O}G_n \overline{\mathbb{O}G_{n+1}} + \mathbb{O}G_{n+1} \overline{\mathbb{O}G_n}) = (e_1, e_2) \tag{3.24}$$

where e_1, e_2 and L are follows:

$$e_1 = \frac{A^2\alpha^{2n}(1+\alpha^2+\alpha^4+\cdots+\alpha^{14}) + B^2\beta^{2n}(1+\beta^2+\beta^4+\cdots+\beta^{14})}{(\alpha-\beta)^2} - L, \tag{3.25}$$

$$e_2 = 2(\sum_{i=0}^{7} w_{n+i} w_{n+1+i}) \tag{3.26}$$

and

$$L = \frac{2AB(-q)^n(a + (-q) + \cdots + (-q)^7)}{p^2+4q} \tag{3.27}$$

where $AB = b^2 - abp - qa^2$.

Proof. It follows from the definition of norm that,

$$Nr(\widehat{\mathbb{O}G_n}) = (\mathbb{O}G_n \overline{\mathbb{O}G_n},\ \ \mathbb{O}G_n \overline{\mathbb{O}G_{n+1}} + \mathbb{O}G_{n+1} \overline{\mathbb{O}G_n}).$$

To get the desired results we first calculate e_1 as follows:

$$e_1 = \frac{A^2\alpha^{2n}(1+\alpha^2+\alpha^4+\cdots+\alpha^{14})}{(\alpha-\beta)^2} + \frac{B^2\beta^{2n}(1+\beta^2+\beta^4+\cdots+\beta^{14})}{(\alpha-\beta)^2} - \frac{2AB(-q)^n(a+(-q)+\cdots+(-q)^7)}{(\alpha-\beta)^2}. \tag{3.28}$$

The dual part of the equation (3.24) can easily be calculated because of the property $\varepsilon^2 = 0$. After direct calculations, the following relation is obtained.

$$e_2 = \mathbb{O}G_n \overline{\mathbb{O}G_{n+1}} + \mathbb{O}G_{n+1} \overline{\mathbb{O}G_n} = 2\Big(\sum_{i=0}^{7} w_{n+i} w_{n+1+i}\Big). \tag{3.29}$$

Q.E.D.

Norm values of dual Fibonacci and Lucas octonions are studied in [6,13]. Utilizing the equation (3.24) together with the appropriate initial values one can get the following equation which is identical to the norm of dual Fibonacci octonions is given in [6]

$$Nr(\widehat{\mathbb{O}_n}) = 21(F_{2n+7} + 2F_{2n+8}\varepsilon). \tag{3.30}$$

In order to see that 3.24 contains real Horadam quaternions (see [7]) one can calculate the $\underline{\alpha}$, $\underline{\beta}$ and use real part of the equation.

4 Conclusion

This paper generalizes similar papers in this area up to now on dual quaternions and octonions. Thanks to this generalization the important properties concerning all the dual octonions such as the Binet formula, the generating function, the Cassini, d'Ocagne's and Vadja identities can be easily obtained.

References

[1] J. Baez, *The octonions*, Bulletin of the American Mathematical Society **(39)2** (2002) 145–205.

[2] P. Catarino and P. Vasco, *On dual k-pell quaternions and octonions*, Mediterranean Journal of Mathematics, **(14)2** (2017) 75.

[3] J. H. Conway and D. A. Smith, *On quaternions and octonions*, AK Peters/CRC Press (2003).

[4] L. E. Dickson, *On quaternions and their generalization and the history of the eight square theorem*, Annals of Mathematics (1919) 155–171.

[5] C. Flaut and V. Shpakivskyi, *Some identities in algebras obtained by the cayley-dickson process*, Advances in Applied Clifford Algebras **(23)1** (2013) 63–76.

[6] S. Halıcı, *On dual fibonacci octonions*, Advances in Applied Clifford Algebras **(25)4** (2015) 905–914.

[7] S. Halıcı and A. Karataş, *On a generalization for fibonacci quaternions*, Chaos, Solitons and Fractals **98** (2017) 178–182.

[8] P. Haukkanen, *A note on horadam's sequence*, Fibonacci Quart. **(40)4** (2002) 358–361.

[9] A. F. Horadam, *Basic properties of a certain generalized sequence of numbers.*, Fibonacci Quart. **(3)3** (1965) 161–176.

[10] A. F. Horadam, *Special properties of the sequence wn (a, b; p, q)*, Fibonacci Quart. **(5)5** (1967) 424–434.

[11] A. Karataş and S. Halici, *Horadam octonions*, Analele Universitatii" Ovidius" Constanta-Seria Matematica **(25)3** (2017) 97–106.

[12] M. N. S. Swamy, *On generalized fibonacci quaternions*, Fibonacci Quart. **(11)5** (1973) 547–549.

[13] Z. Ű nal, Ű. Tokeser and G. Bilgici, *Some properties of dual fibonacci and dual lucas octonions*, Advances in Applied Clifford Algebras **27.2** (2016) 1907–1916.